Well-posedness of Half-Harmonic Map Heat Flows for Rough Initial Data

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Why half-harmonic?

From Plateau flow to half-harmonic map heat flow

- Plateau flow on $B^n \subset \mathbb{R}^n$ with boundary data $u : \partial B^n \to \mathbb{S}^{m-1}$.

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- Energy minimization via harmonic (Poisson) extension $\bar{u}$ of u:

$$E_{1/2}(u) = \frac{1}{2}\int_{B^n} |\nabla \bar{u}|^2 = \frac{c}{2}\int_{\partial B^n}\int_{\partial B^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+1}},$$

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$$(-\Delta)^{1/2} u = \partial_\nu \bar{u}|_{\partial B^n}.$$

- Boundary equation $\partial_t u + d\pi(u)\, \partial_\nu \bar{u} = 0$ becomes

$$\boxed{\partial_t u + d\pi(u)(-\Delta)^{1/2} u = 0}$$

i.e. the half-harmonic map heat flow (gradient flow of $E_{1/2}$).

Further motivation

- The Haldane–Shastry Hamiltonian

$$H = \sum_{i<j}^{L} \frac{1}{\sin^2\left(\frac{\pi}{L}(i-j)\right)} \frac{1 - \vec{\sigma}_i \cdot \vec{\sigma}_j}{2}$$

  converges in the continuum to the half-Laplacian $(-\Delta)^{1/2}$ (suggesting the Hamilton flow)[1].
- The BV norm is comparable (under scaling) to the $\frac{1}{2}$-energy, so $E_{1/2}$ is a natural BV-replacement, e.g. in flows[2].
- Fractional flows are also applied in image processing (nonlocal denoising and inpainting)[3].

[1] E. Lenzmann, A. Schikorra, arXiv:1702.05995, [2] J.W. Barrett, X. Feng, A. Prohl, arXiv:0712.2528, [3] J.A. Iglesias, G. Mercier, arXiv:2201.13281

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- Half Version:

$$\partial_t u = -\, d\pi(u)\, \nabla_{L^2} E_{1/2}(u) \;=\; -(-\Delta)^{1/2} u + u|d_{1/2}u|^2_{od},$$

Scaling and critical spaces

- **Scaling:** HHMHF is invariant under

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  - Bounded mean oscillation $\mathrm{BMO}(\mathbb{R}^n)$.
- In 1D: $H^{\frac{1}{2}}$ is critical, since $\frac{n}{2} = \frac{1}{2}$.

Known results in critical spaces

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with non-increasing 1/2–Dirichlet energy.

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  - Regularity: $u \in L^\infty_t H^{1/2}_x \cap H^1_t L^2_x$ and

    $$u \in C^\infty((T_k, T_{k+1}); N)$$

    except at finitely many times $0 < T_1 < \cdots < T_n < \infty$.

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    $$u \in C^\infty((T_k, T_{k+1}); N)$$
    except at finitely many times $0 < T_1 < \cdots < T_n < \infty$.
  - Bubble bound: the number of singular times satisfies
    $$n(u_0) \ \leq \ \frac{E_{1/2}(u_0)}{\varepsilon_0},$$
    where $\varepsilon_0 > 0$ is the minimal 1/2–energy of a non-constant half-harmonic map.

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  - As $t \to \infty$: convergence to a half-harmonic limit map modulo bubbling.

Base Idea: Koch-Tartaru, Carelson measures, Half-space

Carleson Measures

- A measure μ on $\mathbb{R}^{n+1}_+$ is Carleson if

$$\sup_{Q \subset \mathbb{R}^n} \frac{\mu(T(Q))}{|Q|} < \infty, \qquad T(Q) = Q \times (0, l(Q)].$$

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- Characterisation: $f \in \mathrm{BMO}(\mathbb{R}^n)$ iff

$$d\mu(x,t) = t\,|\nabla(\Phi_t * f)(x)|^2\,dx\,dt$$

is Carleson, where $\Phi(x) = \pi^{-n/2}e^{-|x|^2}$ (Gaussian heat kernel), $\Phi_t(x) = t^{-n}\Phi(x/t)$.

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$$\|u\|_X = \sup_{t>0} t^{1/2}\|u(t)\|_{L^\infty} + \sup_{x,R}\Big(\frac{1}{|B(x,R)|}\int_0^{R^2}\int_{B(x,R)} |u|^2 \, dy \, dt\Big)^{1/2}.$$

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- Initial data (rough):

$$\|u_0\|_{\mathrm{BMO}^{-1}} := \sup_{x,R} \Big(\frac{1}{|B(x,R)|} \int_0^{R^2} \int_{B(x,R)} |e^{t\Delta} u_0|^2 \, dy \, dt \Big)^{1/2}.$$

Main results

Theorem 1 (Existence)

There exists $\varepsilon > 0$ such that for all $a \in A$ with $[a]_{A_T} < \varepsilon$ and $|a|^2 = 1$, there exists $u \in X_T$ satisfying $|u|^2 = 1$ and

$$\begin{cases} \partial_t u = d\pi_{\mathbb{S}^{n-1}}(-\Delta)^{1/2}u, \\ u(0) = a, \end{cases}$$

where the initial condition is attained in the sense of Schwartz distributions.

In particular: for all $a \in V$ there exists some $T > 0$ and a solution $u \in X_{0,T}$ with $u(0) = a$.

Theorem 2 (Uniqueness)

There exists $\delta > 0$ such that solutions $u_1, u_2 \in X_T$ to (9) with $u_1(0) = u_2(0) \in A$ and

$$\limsup_{R \downarrow 0} [u_i]_{X_R} \le \delta, \quad i = 1, 2,$$

satisfy $u_1 \equiv u_2$ in X_T.

In particular: uniqueness holds for solutions in $X_{0,T}$ and for small solutions in X_T as constructed in Theorem 1. Theorem 2 only requires smallness near $t = 0$.

Theorem 3 (Continuous dependence)

Let $a \in V$ and $(a_n) \subset V$ with

$$\lim_{n\to\infty}\left(\|a_n - a\|_{L^\infty} + [a_n - a]_{A_\infty}\right) = 0.$$

Then there exists $T > 0$ such that the corresponding unique solutions $u_n, u \in X_{0,T}$ satisfy

$$\lim_{n\to\infty}\left(\|u_n - u\|_{L^\infty} + [u_n - u]_{X_T^{(0)}}\right) = 0.$$

Theorem 4 (Q_0–framework)

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- Embeddings:

$$\dot{B}^{n/2}_{2,\infty}(\mathbb{R}^n) \; \subset \; Q_0(\mathbb{R}^n), \quad BV(\mathbb{R}) \subset Q_0(\mathbb{R}),$$

hence Q_0 provides a substitute for BMO in the well-posedness theory.

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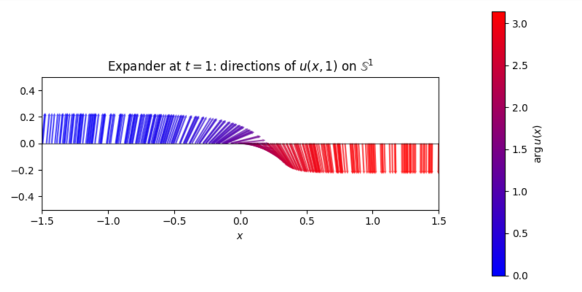
$$a(x) = \varphi\Big(\frac{x}{|x|}\Big)\,, \qquad x \in \mathbb{R}^n \setminus \{0\},$$
- There exists $l = l(n)$ such that if a is of the above homogeneous form and $\mathrm{Lip}(\varphi) \leq l$, then the unique solution u satisfies
$$u(x,t) = u\big(\tfrac{x}{t}, 1\big)\,, \qquad (x,t) \in \mathbb{R}^n \times [0,\infty).$$

Rough data and self similar expander

- Simulated example with jump initial data:

$$u_0(x) \;=\; e_2\,\chi_{(-\infty,0)}(x) \;-\; e_2\,\chi_{(0,\infty)}(x).$$



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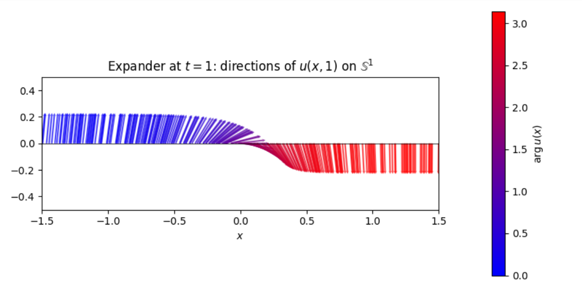
- Self-similar S^1-valued expander on $\mathbb{R}$:

$$u(x,t) \;\approx\; e^{-i\,\arctan(x/t)}.$$



Proof ideas

Proof of Theorem 1 (Existence) — I

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- Remedy: decompose

$$d_{1/2} p \; = \; d_{1/2}^{(0,1)} p + d_{1/2}^{(1,\infty)} p,$$

where $d_{1/2}^{(0,1)} p := d_{1/2} p \cdot \mathbf{1}_{|x-y|<1}$ (local), $d_{1/2}^{(1,\infty)} p := d_{1/2} p \cdot \mathbf{1}_{|x-y|\geq 1}$ (nonlocal).

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- **Spherical values** Consider $v = |u|^2 - 1$

$$\partial_t v + (-\Delta)^{1/2} v = 2v\,|d_{1/2}u|^2_{od}, \qquad v(t_0) = 0.$$

Proof of Theorem 4 (Idea)

- Translation and scaling invariance $\Rightarrow$ reduce to

$$\int_0^1 \fint_{B_1(0)} |d_{1/2}(p_t * a)|_{od}^2 \, dx \, dt \ \leq c \, [a]_{Q_0}^2.$$

Proof of Theorem 4 (Idea)

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- Local part $(0, t)$ estimated like Carleson measures $\sim$ BMO argument.

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- $$(*)_{(t,\infty)} \;\leq\; I_1 + I_2,$$

  with

$$I_1 = \int_{B_1(0)} \int_{B_1(0)} \frac{|a(x+h) - a(x)|^2}{|h|^n} \, dh \, dx,$$

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- Bounds: $I_1 \le c\,[a]^2_{Q_0}, \quad I_2 \le c\,[a]^2_{\mathrm{BMO}} \le c\,[a]^2_{Q_0}.$

Outlook

Outlook — Shrinkers & Thresholding

- **Shrinkers (finite-time blow-up candidates):**

$$u(x,t) = U\Big(\frac{x}{T-t}\Big), \qquad T > 0.$$

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- **Shrinkers (finite-time blow-up candidates):**

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- **Thresholding (Laux and Yip type scheme):**

$$u^h(t) = \sum_{k=0}^{N} u^k \, \chi_{(kh,(k+1)h]}(t), \qquad u^{k+1} = \frac{p_h * u^k}{|p_h * u^k|}.$$

Practical use: stable, projection-based timestepping; questions: convergence to HHMHF, error/control of bubbling.

Thank you for your attention!