Well-posedness of Half-Harmonic Map Heat Flows for Rough Initial Data

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joint with Christof Melcher

December 4, 2025

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Funded by the CRC “Sparsity and Singular Structures” (SFB 1481)

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- Existence has already been shown for different kinds of PDEs (e.g., heat flow, harmonic map flow, ...).

Rough initial data and self similar expander

- Simulated example with jump initial data:

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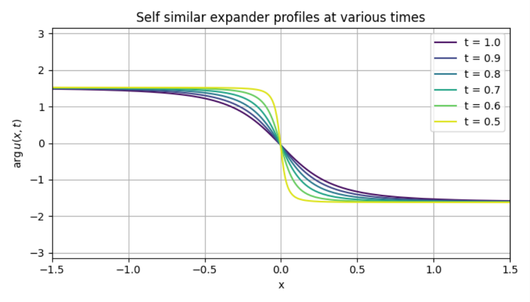

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hence the self-similar form

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  - Bounded mean oscillation $\mathrm{BMO}(\mathbb{R}^n)$.
- In 1D: $\dot{H}^{\frac{1}{2}}$ is critical, since $\frac{n}{2} = \frac{1}{2}$.

Known results in critical spaces

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  - Bubble bound: the number of singular times satisfies
    $$n(u_0) \;\leq\; \frac{E_{1/2}(u_0)}{\varepsilon_0},$$
    where $\varepsilon_0 > 0$ is the minimal 1/2–energy of a non-constant half-harmonic map.

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  - As $t \to \infty$: convergence to a half-harmonic limit map modulo bubbling.

Main Idea: Koch-Tartaru, Maximal function, Half-space

Maximal function

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$$f \in \mathrm{BMO}(\mathbb{R}^n) \iff \|M(f)\|_{L^\infty(\mathbb{R}^n \times (0,\infty))} < \infty,$$

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$$f \in \overline{\mathrm{VMO}}(\mathbb{R}^n) \iff \lim_{R \to 0} \|M(f)\|_{L^\infty(\mathbb{R}^n \times (0,R))} = 0.$$

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$$\|u\|_X = \sup_{t>0} t^{1/2} \|u(t)\|_{L^\infty} + \sup_{x,R} \left(\frac{1}{|B(x,R)|} \int_0^{R^2} \int_{B(x,R)} |u|^2 \, dy \, dt \right)^{1/2}.$$

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- Initial data (rough):

$$\|u_0\|_{\mathrm{BMO}^{-1}} := \sup_{x,R}\Big(\frac{1}{|B(x,R)|}\int_0^{R^2}\int_{B(x,R)} |e^{t\Delta}u_0|^2\,dy\,dt\Big)^{1/2}.$$

Main results

Theorem 1 (Existence)

There exists $\varepsilon > 0$ such that for all $a \in A$ with $[a]_{A_T} < \varepsilon$ and $|a|^2 = 1$, there exists $u \in X_T$ satisfying $|u|^2 = 1$ and

$$\begin{cases} \partial_t u = -d\pi_{\mathbb{S}^{n-1}}(u)(-\Delta)^{1/2}u, \\ u(0) = a, \end{cases}$$

where the initial condition is attained in the sense of Schwartz distributions.

In particular: for all $a \in V$ there exists some $T > 0$ and a solution $u \in X_{0,T}$ with $u(0) = a$.

Theorem 2 (Uniqueness)

There exists $\delta > 0$ such that solutions $u_1, u_2 \in X_T$ to (9) with $u_1(0) = u_2(0) \in A$ and

$$\limsup_{R \downarrow 0} [u_i]_{X_R} \leq \delta, \quad i = 1, 2,$$

satisfy $u_1 \equiv u_2$ in X_T.

In particular: uniqueness holds for solutions in $X_{0,T}$ and for small solutions in X_T as constructed in Theorem 1. Theorem 2 only requires smallness near $t = 0$.

Theorem 3 (Continuous dependence)

Let $a \in V$ and $(a_n) \subset V$ with

$$\lim_{n\to\infty} \left(\|a_n - a\|_{L^\infty} + [a_n - a]_{A_\infty} \right) = 0.$$

Then there exists $T > 0$ such that the corresponding unique solutions $u_n, u \in X_{0,T}$ satisfy

$$\lim_{n\to\infty} \left(\|u_n - u\|_{L^\infty} + [u_n - u]_{X_T^{(0)}} \right) = 0.$$

Theorem 4 (Q_0–framework)

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- Thus $Q_0(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n)$ but captures relevant rough data and extends critical Sobolev well-posedness.

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- Thus $Q_0(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n)$ but captures relevant rough data and extends critical Sobolev well-posedness.
- Embeddings:

$$\dot{B}^{n/2}_{2,\infty}(\mathbb{R}^n) \ \subset \ Q_0(\mathbb{R}^n), \quad BV(\mathbb{R}) \subset Q_0(\mathbb{R}),$$

  hence Q_0 provides a substitute for BMO in the well-posedness theory.

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$$u(x,t) = u\left(\tfrac{x}{t}, 1\right), \qquad (x,t) \in \mathbb{R}^n \times [0,\infty).$$

Proof ideas

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- Duhamel formulation:

$$\partial_t u = -(-\Delta)^{1/2} u \ + \ |d_{1/2} u|_{od}^2 \, u.$$

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- Remedy: decompose

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where $d_{1/2}^{(0,1)} p := d_{1/2} p \cdot \mathbf{1}_{|x-y|<1}$ (local), $d_{1/2}^{(1,\infty)} p := d_{1/2} p \cdot \mathbf{1}_{|x-y|\geq 1}$ (nonlocal).

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- **Spherical values** Consider $v = |u|^2 - 1$

$$\partial_t v + (-\Delta)^{1/2} v = 2v\,|d_{1/2}u|^2_{od}, \qquad v(t_0) = 0.$$

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- Translation and scaling invariance $\Rightarrow$ reduce to

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Outlook

Outlook — Shrinkers & Thresholding

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$$u(x,t) = U\Big(\frac{x}{T-t}\Big), \qquad T > 0.$$

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- **Thresholding (Laux and Yip type scheme):**

$$u^h(t) = \sum_{k=0}^{N} u^k\, \chi_{(kh,(k+1)h]}(t), \qquad u^{k+1} = \frac{p_h * u^k}{|p_h * u^k|}.$$

The End

Thank you for your attention!