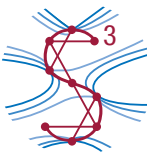
Well-posedness of half-harmonic map heat flows for rough initial data

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joint with Christof Melcher

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2 Main results

3 Koch–Tataru method

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Harmonic map heat flow $\partial_t \boldsymbol{u} = \Delta \boldsymbol{u} + |\nabla \boldsymbol{u}|^2 \boldsymbol{u} \quad (\beta = 0,\, \alpha = 1)$

Navier–Stokes $\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0, \quad \operatorname{div} \boldsymbol{u} = 0$

Harmonic map heat flow

Energy for maps $\boldsymbol{u}\colon \mathbb{R}^n \to \mathbb{S}^m$

$$D(\boldsymbol{u}) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \boldsymbol{u}|^2 \, dx$$

J. Eells, J. H. Sampson, Amer. J. Math. 86 (1964); J.-M. Coron, J.-M. Ghidaglia, C. R. Acad. Sci. Paris 308 (1989)

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$$\partial_t \boldsymbol{u} = -\nabla^{\tan} D(\boldsymbol{u}) = \Delta \boldsymbol{u} + |\nabla \boldsymbol{u}|^2 \boldsymbol{u}$$

Critical dimension $n = 2$, since $D(\boldsymbol{u}(\lambda\,\cdot)) = \lambda^{2-n} D(\boldsymbol{u})$.

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Finite-time blow-up by symmetry (Coron–Ghidaglia).

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Half-harmonic map heat flow

Energy for maps $\boldsymbol{u}\colon \mathbb{R}^n \to \mathbb{S}^m$

$$E(\boldsymbol{u}) = \frac{c_n}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\boldsymbol{u}(x) - \boldsymbol{u}(y)|^2}{|x-y|^{n+1}}\, dy\, dx = \frac{1}{2} \|(-\Delta)^{1/4} \boldsymbol{u}\|_{L^2}^2$$

Integral representation: V. Millot, Y. Sire, Arch. Ration. Mech. Anal. 215 (2015)

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$$\begin{aligned} \partial_t \boldsymbol{u} &= -\nabla^{\tan} E(\boldsymbol{u}) \\ &= -(-\Delta)^{1/2} \boldsymbol{u} + \Big(\frac{c_n}{2} \int_{\mathbb{R}^n} \frac{|\boldsymbol{u}(x) - \boldsymbol{u}(y)|^2}{|x-y|^{n+1}}\, dy \Big)\, \boldsymbol{u}(x) \end{aligned}$$

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Critical dimension $n = 1$, since $E(\boldsymbol{u}(\lambda\,\cdot)) = \lambda^{1-n} E(\boldsymbol{u})$.

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Half gradient

$$d_{1/2}\boldsymbol{u}(x,y) = \sqrt{\frac{c_n}{2}}\,\frac{\boldsymbol{u}(x)-\boldsymbol{u}(y)}{|x-y|^{1/2}}$$

with the off-diagonal inner product

$$\langle \boldsymbol{F},\boldsymbol{G}\rangle_{\mathrm{od}}(x) = \int_{\mathbb{R}^n} \boldsymbol{F}(x,y)\cdot\boldsymbol{G}(x,y)\,\frac{dy}{|x-y|^n}, \qquad |\boldsymbol{F}|^2_{\mathrm{od}} = \langle \boldsymbol{F},\boldsymbol{F}\rangle_{\mathrm{od}}$$

M. Mazowiecka, A. Schikorra, J. Funct. Anal. 275 (2018); flow in this form: J. Wettstein, Nonlinear Anal. 214 (2022)

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Hence the flow reads

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Shrinker $\boldsymbol{u}(x,t) = \boldsymbol{U}\left(\frac{x}{T-t}\right)$ $(-\Delta)^{1/2}\boldsymbol{U}(y) + y \cdot \nabla \boldsymbol{U}(y) = |d_{1/2}\boldsymbol{U}|^2_{\mathrm{od}}(y)\,\boldsymbol{U}(y)$

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Expander $\boldsymbol{u}(x,t) = \boldsymbol{U}\left(\frac{x}{t}\right)$ $(-\Delta)^{1/2}\boldsymbol{U}(y) - y \cdot \nabla \boldsymbol{U}(y) = |d_{1/2}\boldsymbol{U}|^2_{\mathrm{od}}(y)\,\boldsymbol{U}(y)$

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The counterpart to a shrinker is the expander.

Finding expanders

Homogeneous initial data $\boldsymbol{u}_0(\lambda x) = \boldsymbol{u}_0(x)$ for all $\lambda > 0$

Expanders for HMHF and LLG: P. Germain, M. Rupflin, Ann. IHP Anal. Non Linéaire 28 (2011); S. Gutiérrez, A. de Laire, Nonlinearity 32 (2019)

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Simulated example with jump initial data:

$$\boldsymbol{u}_0(x) \approx \boldsymbol{e}_2\, \chi_{(-\infty,0)}(x) - \boldsymbol{e}_2\, \chi_{(0,\infty)}(x)$$

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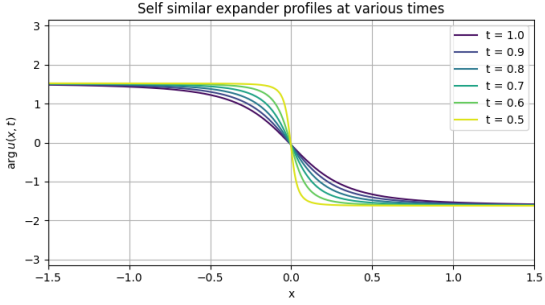
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Self-similar $\mathbb{S}^1$-valued expander on $\mathbb{R}$:

$$\boldsymbol{u}(x,t) = \boldsymbol{U}\left(\tfrac{x}{t}\right)$$



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Main theorem: existence

Theorem

There exists $\varepsilon > 0$ such that for all $\boldsymbol{a}\colon \mathbb{R}^n \to \mathbb{S}^m$ with $\boldsymbol{a} \in \mathcal{A}$ and $[\boldsymbol{a}]_{\mathcal{A}_T} < \varepsilon$, there exists $\boldsymbol{u} \in X_T$ solving

$$\begin{cases} \partial_t \boldsymbol{u} + (-\Delta)^{1/2} \boldsymbol{u} = |d_{1/2} \boldsymbol{u}|^2_{\mathrm{od}}\, \boldsymbol{u}, \\ \boldsymbol{u}(0) = \boldsymbol{a}. \end{cases}$$

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Especially, $\quad L^\infty \cap \mathrm{BV}(\mathbb{R}) \subset \mathcal{A}, \qquad L^\infty \cap \dot{B}^{\frac{n}{2}}_{2,\infty}(\mathbb{R}^n) \subset \mathcal{A}.$

K. Koch, C. Melcher, arXiv:2504.06933

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$\dot{B}^{1/2}_{2,\infty}(\mathbb{R}) \not\subset \dot{H}^{1/2}(\mathbb{R}), \qquad \mathrm{BV}(\mathbb{R}) \not\subset \dot{H}^{1/2}(\mathbb{R}).$

K. Koch, C. Melcher, arXiv:2504.06933

Main theorem: uniqueness

Theorem

There exists $\delta > 0$ such that solutions $\boldsymbol{u}_1, \boldsymbol{u}_2 \in X_T$ with $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) \in \mathcal{A}$ and

$$\limsup_{R \searrow 0} [\boldsymbol{u}_i]_{X_R} \leq \delta, \qquad i = 1, 2,$$

satisfy $\boldsymbol{u}_1 \equiv \boldsymbol{u}_2$.

K. Koch, C. Melcher, arXiv:2504.06933

Borderline regularity of homogeneous initial maps

$$\boldsymbol{a}(x) = \varphi\Big(\frac{x}{|x|}\Big) \qquad \text{for } \varphi\colon \mathbb{S}^{n-1} \to \mathbb{S}^m \text{ Lipschitz}$$

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$$\boldsymbol{a}(x) = \varphi\Big(\frac{x}{|x|}\Big) \qquad \text{for } \varphi\colon \mathbb{S}^{n-1} \to \mathbb{S}^m \text{ Lipschitz}$$

Homogeneity of the Fourier transform implies for $\hat{\boldsymbol{e}}(x) = \frac{x}{|x|}$:

$$\mathcal{F}(\hat{\boldsymbol{e}})(\xi) \sim \frac{\xi}{|\xi|^{n+1}}, \qquad \text{so} \qquad [\hat{\boldsymbol{e}}]^2_{\dot{H}^{n/2}} \sim \int_{\mathbb{R}^n} |\xi|^n \, \frac{d\xi}{|\xi|^{2n}} = \infty$$

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Littlewood–Paley decomposition, Besov regularity:

$$\int_{\mathbb{R}^n} |\xi|^n \frac{d\xi}{|\xi|^{2n}} \sim \sum_{k\in\mathbb{Z}} 2^{kn} \int_{|\xi|\sim 2^k} \frac{d\xi}{|\xi|^{2n}} \geq \sup_{k\in\mathbb{Z}} 2^{kn} \int_{|\xi|\sim 2^k} \frac{d\xi}{|\xi|^{2n}} = [\hat{\boldsymbol{e}}]^2_{\dot{B}^{n/2}_{2,\infty}} < \infty$$

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So $\hat{\boldsymbol{e}} \in L^\infty \cap \dot{B}^{n/2}_{2,\infty}(\mathbb{R}^n) \subset \mathcal{A}$.

K. Koch, C. Melcher, arXiv:2504.06933

Self-similar expanders

We have

$$[\boldsymbol{a}]_{\mathcal{A}} \lesssim \operatorname{Lip}(\boldsymbol{\varphi})\, [\hat{\boldsymbol{e}}]_{\dot{B}^{n/2}_{2,\infty}},$$

K. Koch, C. Melcher, arXiv:2504.06933

Self-similar expanders

We have

$$[\boldsymbol{a}]_{\mathcal{A}} \lesssim \operatorname{Lip}(\varphi)\,[\hat{\boldsymbol{e}}]_{\dot{B}^{n/2}_{2,\infty}},$$

and through a scaling argument, we get that for $\operatorname{Lip}(\varphi) \ll 1$ there exists a self-similar expander

$$\boldsymbol{u}(x,t) = \boldsymbol{U}\left(\tfrac{x}{t}\right) \qquad \text{with} \qquad \boldsymbol{u}(0) = \boldsymbol{a},$$

where the scaling of the half-harmonic map heat flow is $\boldsymbol{u}(\lambda x, \lambda t)$.

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By uniqueness and choosing $\lambda = 1/t$, one obtains

$$\boldsymbol{u}(x,t) = \boldsymbol{u}\big(\tfrac{x}{t}, 1\big), \qquad \text{i.e.} \qquad \boldsymbol{U}(y) = \boldsymbol{u}(y,1).$$

K. Koch, C. Melcher, arXiv:2504.06933

Known results in energy spaces

Theorem (Wettstein, Struwe)

For any $\boldsymbol{u}_0 \in H^{1/2}(\mathbb{S}^1; N)$ *there exists a weak solution*

$$\boldsymbol{u} \colon [0, \infty) \times \mathbb{S}^1 \to N$$

with non-increasing $1/2$*–Dirichlet energy.*

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with non-increasing $1/2$*–Dirichlet energy.*
Regularity: $\boldsymbol{u} \in L^\infty_t H^{1/2}_x \cap H^1_t L^2_x$*, and* $\boldsymbol{u}$ *is smooth on* $(0, \infty) \times \mathbb{S}^1$ *except in finitely many points* (T_k, x_k)*,* $k = 1, \dots, K$*.*

J. Wettstein, Math. Eng. 5 (2023); M. Struwe, Anal. PDE 17 (2024)

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Proof idea: Koch–Tataru method

Harmonic extension $\boldsymbol{v}(t) = p_t * \boldsymbol{a}$: $\quad \partial_t \boldsymbol{v} + (-\Delta)^{1/2} \boldsymbol{v} = 0, \ \boldsymbol{v}(0) = \boldsymbol{a}$

$$\mathcal{M}_T \boldsymbol{a}(x) := \sup_{0<r<T} \Big(\int_0^r \fint_{B_r(x)} |d_{1/2} \boldsymbol{v}|_{\mathrm{od}}^2 \, dy \, dt \Big)^{1/2}$$

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Harmonic map heat flow (analogous to C. Wang) $\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} = 0,\ \boldsymbol{u}(0) = \boldsymbol{a}$

$$\sup_{0<r<\sqrt{T}} \sup_{x\in\mathbb{R}^n} \int_0^{r^2} \fint_{B_r(x)} |\nabla \boldsymbol{u}|^2\, dy\, dt$$

$T = \infty$: equivalent to $\|\boldsymbol{a}\|^2_{\mathrm{BMO}}$ (Carleson characterization)

H. Koch, D. Tataru, Adv. Math. 157 (2001); C. Wang, Arch. Ration. Mech. Anal. 200 (2011); Carleson characterization: E. M. Stein, Harmonic Analysis (1993)

Solution space X_T

Following Koch–Tataru:

$$[\boldsymbol{u}]_{X_T^{(0)}} := \sup_{0<t<T,\, x\in\mathbb{R}^n} t^{\frac{1}{2}} \, |d_{1/2}\boldsymbol{u}|_{\mathrm{od}}(x,t) + \sup_{0<t<T,\, x\in\mathbb{R}^n} \Big(\int_0^t \fint_{B_t(x)} |d_{1/2}\boldsymbol{u}|_{\mathrm{od}}^2(y,s)\, dy\, ds \Big)^{1/2}$$

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Through scaling, we define the higher order seminorm $[\boldsymbol{u}]_{X_T^{(1)}}$, which leads to

$$\|\!|\boldsymbol{u}|\!\|_{X_T} := \sup_{0<t<T,\, x\in\mathbb{R}^n} |\boldsymbol{u}(x,t)| + [\boldsymbol{u}]_{X_T^{(0)}} + [\boldsymbol{u}]_{X_T^{(1)}}$$

H. Koch, D. Tataru, Adv. Math. 157 (2001); K. Koch, C. Melcher, arXiv:2504.06933

Why not $(-\Delta)^{1/4}$?

$$\mathcal{M}_T^* \boldsymbol{a}(x) := \sup_{0<r<T} \Big(\int_0^r \fint_{B_r(x)} |(-\Delta)^{1/4} \boldsymbol{v}|^2 \, dy \, dt \Big)^{1/2}$$

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$$(-\Delta)^{1/4}(\boldsymbol{u} \cdot \boldsymbol{v}) = (-\Delta)^{1/4}\boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot (-\Delta)^{1/4}\boldsymbol{v} - 2 \langle d_{1/4}\boldsymbol{u}, d_{1/4}\boldsymbol{v} \rangle_{\mathrm{od}}$$

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vs. $$d_{1/2}(\boldsymbol{u} \cdot \boldsymbol{v})(x,y) = d_{1/2}\boldsymbol{u}(x,y) \cdot \boldsymbol{v}(x) + \boldsymbol{u}(y) \cdot d_{1/2}\boldsymbol{v}(x,y)$$

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With X_T^* one needs a localized equivalence:

$$\int_0^t \fint_{B_t(x)} |(-\Delta)^{1/4}\boldsymbol{u}|^2 \;\sim\; \int_0^t \fint_{B_t(x)} |d_{1/2}\boldsymbol{u}|^2_{\mathrm{od}}$$

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1 Blow-up and expanders

2 Main results

3 Koch–Tataru method

4 Scaling invariant estimates

5 Outlook

Mild formulation

The mild formulation of the HHMHF is

$$\boldsymbol{u}(t) = S(t)\boldsymbol{a} + G\big(\boldsymbol{u}\,|d_{1/2}\boldsymbol{u}|^2_{\mathrm{od}}\big)(t),$$

$$S(t)\boldsymbol{a} = p_t * \boldsymbol{a}, \qquad G(\boldsymbol{f})(t) = \int_0^t S(t-s)\boldsymbol{f}(s)\,ds.$$

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Goal: find a fixed point in X_T via Picard iteration of

$$\Phi(\boldsymbol{u}) := S(t)\boldsymbol{a} + G\big(\varphi(\boldsymbol{u})\,|d_{1/2}\boldsymbol{u}|^2_{\mathrm{od}}\big),$$

$$\varphi \in C_c^\infty, \qquad \varphi(y) = y \ \text{ for } |y| \le \tfrac{3}{2}, \qquad \varphi(y) = 0 \ \text{ for } |y| \ge 2.$$

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Picard iteration

For the Picard iteration $\boldsymbol{u}_0 = \boldsymbol{a}$, $\boldsymbol{u}_{l+1} = \Phi(\boldsymbol{u}_l)$ we need

$$[S(t)\boldsymbol{a}]_{X_T} \lesssim [\boldsymbol{a}]_{\mathcal{A}_T}$$

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such that

$$\|\!|G\big(\varphi(\boldsymbol{u})\,|d_{1/2}\boldsymbol{u}|^2_{\mathrm{od}}\big)|\!\|_{X_T} \lesssim [\boldsymbol{u}]^2_{X_T}.$$

Homogeneous estimate

First, the homogeneous estimate holds by construction of $\mathcal{A}$, through the following estimate:

$$\sup_{0<t<T} t \, \|d_{1/2}(p_t * \boldsymbol{a})\|^2_{L^\infty_{\mathrm{od}}} \lesssim [\boldsymbol{a}]^2_{\mathcal{A}_T}$$

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By scaling and translation it suffices to show

$$|d_{1/2}(p_1 * \boldsymbol{a})|_{\mathrm{od}}(0) \lesssim \Big(\sup_{x\in\mathbb{R}^n} \int_0^1 \fint_{B_1(x)} |d_{1/2}(p_s * \boldsymbol{a})|^2_{\mathrm{od}} \, dy \, ds \Big)^{1/2}$$

Homogeneous estimate

With $\boldsymbol{v}(s) = p_s * \boldsymbol{a}$:

$$\boldsymbol{v}(1) = p_1 * \boldsymbol{a} = p_{1-s} * \boldsymbol{v}(s), \qquad s \in (0, \tfrac{1}{2}]$$

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Averaging over $s \in (0, \frac{1}{2}]$:

$$|d_{1/2}\boldsymbol{v}|_{\mathrm{od}}(0,1) \lesssim \int_0^1 \int_{\mathbb{R}^n} \frac{|d_{1/2}\boldsymbol{v}(s)|_{\mathrm{od}}(y)}{1+|y|^{n+1}}\, dy\, ds \lesssim \Big(\sup_x \int_0^1 \fint_{B_1(x)} |d_{1/2}\boldsymbol{v}|_{\mathrm{od}}^2\, dy\, ds \Big)^{1/2}$$

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Goal: $\|\!|G(\boldsymbol{f})|\!\|_{X_T} \lesssim \|\!|\boldsymbol{f}|\!\|_{Y_T}$

Local and nonlocal part of $d_{1/2}$

To estimate $\|d_{1/2}G(\boldsymbol{f})\|_{L^\infty_{\mathrm{od}}}$, one would want $|d_{1/2}p|_{\mathrm{od}} \in L^1$.

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Solution: split $d_{1/2}$ into local and nonlocal parts, i.e.

$$|d_{1/2}G(\boldsymbol{f})|_{\mathrm{od}} \leq |d_{1/2}G(\boldsymbol{f})|^{(0,1)}_{\mathrm{od}} + |d_{1/2}G(\boldsymbol{f})|^{(1,\infty)}_{\mathrm{od}}$$

Local and nonlocal part of $d_{1/2}$

where

$$\left(|d_{1/2}\boldsymbol{g}|_{\mathrm{od}}^{(r_1,r_2)}\right)^2(x) := \frac{c_n}{2}\int_{B_{r_2}\setminus B_{r_1}} \frac{|\boldsymbol{g}(x+h)-\boldsymbol{g}(x)|^2}{|h|^{n+1}}\,dh.$$

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$$|d_{1/2}G(\boldsymbol{f})|_{\mathrm{od}}^{(1,\infty)}(x) = \left(\frac{c_n}{2}\int_{|h|\geq 1} \frac{|G(\boldsymbol{f})(x+h)-G(\boldsymbol{f})(x)|^2}{|h|^{n+1}}\,dh\right)^{1/2} \leq C\,\|G(\boldsymbol{f})\|_{L^\infty},$$

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and $|d_{1/2}G(\boldsymbol{f})|_{\mathrm{od}}^{(0,1)}$ can be estimated, because $|d_{1/2}p|_{\mathrm{od}}^{(0,1)} \in L^1$.

Localization with a smoothing kernel

$\boldsymbol{w} = G(\boldsymbol{f})$ solves $\partial_t \boldsymbol{w} + (-\Delta)^{1/2} \boldsymbol{w} = \boldsymbol{f}$.

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i.e. a pseudolocal structure, we get

$$\int_0^1 \int_{B_1} |d_{1/2}\boldsymbol{w}|^2_{\mathrm{od}}\, dx\, dt \lesssim \|\boldsymbol{f}\|^2_{Y_1}.$$

Modified equation

Through the Picard iteration we get a mild solution $\boldsymbol{u}$ to

$$\begin{cases} \partial_t \boldsymbol{u} + (-\Delta)^{1/2} \boldsymbol{u} = \varphi(\boldsymbol{u}) \, |d_{1/2} \boldsymbol{u}|^2_{\mathrm{od}}, \\ \boldsymbol{u}(0) = \boldsymbol{a}. \end{cases}$$

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It is missing that $|\boldsymbol{u}|^2 = 1$.

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But $\rho = |\boldsymbol{u}|^2 - 1$ satisfies the linear equation

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which guarantees uniqueness in the class $L^\infty((0,T) \times \mathbb{R}^n)$. Indeed:

$$\begin{aligned} \sup_{t \in [0,T]} \|\rho_1(t) - \rho_2(t)\|_{L^\infty} &= \sup_{t \in [0,T]} \left\| G\big((\rho_1 - \rho_2)\, |d_{1/2} \boldsymbol{u}|^2_{\mathrm{od}}\big)(t) \right\|_{L^\infty} \\ &\lesssim \sup_{t \in [0,T]} \|\rho_1(t) - \rho_2(t)\|_{L^\infty}\, [\boldsymbol{u}]^2_{X_T} \end{aligned}$$

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so $\rho = 0$.

The space Q_0

For a measurable $\boldsymbol{a}$ we define

$$[\boldsymbol{a}]_{Q_0}^2 := \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{r^n} \int_{B_r(x)} \int_{B_r(x)} \frac{|\boldsymbol{a}(y) - \boldsymbol{a}(z)|^2}{|y-z|^n}\, dy\, dz$$

M. Essén, S. Janson, L. Peng, J. Xiao, Indiana Univ. Math. J. 49 (2000); BV: A. Cohen, W. Dahmen, I. Daubechies, R. DeVore, Rev. Mat. Iberoam. 19 (2003)

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$$Q_0 \subset \mathrm{BMO}, \qquad \mathrm{BV}(\mathbb{R}) \subset Q_0, \qquad \dot{B}^{n/2}_{2,\infty}(\mathbb{R}^n) \subset Q_0$$

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We can show $L^\infty \cap Q_0 \subset \mathcal{A}$.

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$Q_0 \subset \mathcal{A}$: proof idea

We split again into local $(0, t)$ and nonlocal (t, ∞):

$$|d_{1/2}(p_t * \boldsymbol{a})|_{\mathrm{od}} \leq |d_{1/2}(p_t * \boldsymbol{a})|_{\mathrm{od}}^{(0,t)} + |d_{1/2}(p_t * \boldsymbol{a})|_{\mathrm{od}}^{(t,\infty)}$$

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The local part behaves like BMO:

$$\int_0^1 \fint_{B_1} \left(|d_{1/2}(p_t * \boldsymbol{a})|_{\mathrm{od}}^{(0,t)}\right)^2 \lesssim [\boldsymbol{a}]_{\mathrm{BMO}}^2 \lesssim [\boldsymbol{a}]_{Q_0}^2$$

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The nonlocal part behaves like Q_0:

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Uniqueness

Thus we have existence for rough initial data. Next we need uniqueness.

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Theorem (Uniqueness)

There exists $\delta > 0$ *such that solutions* $\boldsymbol{u}_1, \boldsymbol{u}_2 \in X_T$ *with* $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) \in \mathcal{A}$ *and*

$$\limsup_{R \searrow 0} [\boldsymbol{u}_i]_{X_R} \leq \delta, \qquad i = 1, 2,$$

satisfy $\boldsymbol{u}_1 \equiv \boldsymbol{u}_2$.

Uniqueness: open and closed

Show that

$$I := \{t \in [0, T] : \boldsymbol{u}_1 = \boldsymbol{u}_2 \text{ on } [0, t] \times \mathbb{R}^n\}$$

is open and closed. Closed by continuity.

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Open: for $t_0 > 0$ the localized seminorms become small, e.g.

$$\sup_{t_0 < t < R,\, x} (t - t_0)^{1/2} \, |d_{1/2} \boldsymbol{u}|_{\mathrm{od}}(x, t) \leq \left(\frac{R - t_0}{t_0}\right)^{1/2} [\boldsymbol{u}]_{X_T} \xrightarrow{R \searrow t_0} 0,$$

Uniqueness: open and closed

Show that

$$I := \{t \in [0, T] : \boldsymbol{u}_1 = \boldsymbol{u}_2 \text{ on } [0, t] \times \mathbb{R}^n\}$$

is open and closed. Closed by continuity.

Open: for $t_0 > 0$ the localized seminorms become small, e.g.

$$\sup_{t_0 < t < R,\, x} (t - t_0)^{1/2} \, |d_{1/2} \boldsymbol{u}|_{\mathrm{od}}(x, t) \leq \left(\frac{R - t_0}{t_0}\right)^{1/2} [\boldsymbol{u}]_{X_T} \xrightarrow{R \searrow t_0} 0,$$

Thus, without loss of generality, let $t_0 = 0$.

Uniqueness: $t_0 = 0$

For $t_0 = 0$ we have $[\boldsymbol{u}_i]_{X_R} < 2\delta$ for small R, since $\limsup_{R \searrow 0} [\boldsymbol{u}_i]_{X_R} \leq \delta$.

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Difference estimate:

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{X_R} \leq C\big([\boldsymbol{u}_1]_{X_R}, [\boldsymbol{u}_2]_{X_R}\big) \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{X_R}$$

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Smallness: $C(r, r) < \frac{1}{2}$ for $r < 2\delta$, hence

$$\|\!|\boldsymbol{u}_1 - \boldsymbol{u}_2|\!\|_{X_R} \leq \tfrac{1}{2}\, \|\!|\boldsymbol{u}_1 - \boldsymbol{u}_2|\!\|_{X_R} \quad \Longrightarrow \quad \boldsymbol{u}_1 = \boldsymbol{u}_2 \text{ on } [0, R].$$

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Outlook

Shrinkers in higher dimensions: find shrinkers for $n > 1$,

$$\boldsymbol{u}(x,t) = \boldsymbol{U}\big(\tfrac{x}{T-t}\big), \qquad (-\Delta)^{1/2}\boldsymbol{U}(y) + y \cdot \nabla \boldsymbol{U}(y) = |d_{1/2}\boldsymbol{U}|^2_{\mathrm{od}}(y)\, \boldsymbol{U}(y)$$

Blow-up: J. Wettstein, Math. Eng. 5 (2023); M. Struwe, Anal. PDE 17 (2024); infinite-time blow-up: Y. Sire, J. Wei, Y. Zheng, Amer. J. Math. 143 (2021)

Outlook

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Blow-up in the critical dimension $n = 1$: Show existence or non-existence of blow-ups, i.e. (x_0, T) such that

$$\limsup_{t \nearrow T} \int_{B_R(x_0)} |(-\Delta)^{1/4}\boldsymbol{u}(t)|^2\, dx \geq \varepsilon_1 \qquad \text{for all } R > 0,$$

for a fixed $\varepsilon_1 > 0$.

Blow-up: J. Wettstein, Math. Eng. 5 (2023); M. Struwe, Anal. PDE 17 (2024); infinite-time blow-up: Y. Sire, J. Wei, Y. Zheng, Amer. J. Math. 143 (2021)

Thank you for listening!



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