Thresholding Scheme for the Half-Harmonic Map Heat Flow

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The Half-Harmonic Map Heat Flow

- Half-Dirichlet energy for maps $u : \mathbb{T}^d \to \mathbb{S}^m$:

$$E_{1/2}(u) = \frac{c_d}{4} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+1}} \, dy \, dx = \frac{1}{2} \|(-\Delta)^{1/4} u\|_{L^2}^2$$
$$=: \frac{1}{2} \int_{\mathbb{T}^d} |d_{1/2} u|_{\mathrm{od}}^2 \, dx.^1$$

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- **Gradient flow** of $E_{1/2}$ on $\mathbb{S}^m$, $u(0) = u^0 : \mathbb{T}^d \to \mathbb{S}^m$:

$$\partial_t u = -d\pi_{\mathbb{S}^m}(u) \nabla_{L^2} E_{1/2} = -(-\Delta)^{1/2} u + |d_{1/2} u|_{\mathrm{od}}^2 \, u.$$

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- $(-\Delta)^{1/2}$ defined via Fourier coefficients:

$$\widehat{(-\Delta)^{1/2} u}(k) = |k| \, \hat{u}(k), \quad k \in \mathbb{Z}^d.$$

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The Thresholding Scheme

Motivated by Laux–Yip [LY19].

- Solve the **half-heat equation** $\partial_t v + (-\Delta)^{1/2} v = 0$, $v(0) = u^0$. The solution is given by

$$v(t,x) = (P_t * u^0)(x), \quad \text{where } \hat{P}_t(k) = e^{-t|k|},\ k \in \mathbb{Z}^d.$$

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- **Algorithm:** Given time step $h > 0$, iterate:

$$u^{n+1} = \frac{P_h * u^n}{|P_h * u^n|}, \qquad u^h(t) := \sum_k u^k \, \chi_{[kh,\,(k+1)h)}(t).$$

From MCF to Harmonic Map Flows

Flow	$\hat{P}_h(k)$	Ref
MCF (codim. 1)	$e^{-h\|k\|^2}$	[MBO92; LO16]
MCF (codim. 2) + HMHF	$e^{-h\|k\|^2}$	[LY19]
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Key adaptation: replace the Gaussian $e^{-h|k|^2}$ by the Poisson kernel $e^{-h|k|}$.

Weak Solution

Definition

$u \in L^\infty(0,T; H^{1/2}(\mathbb{T}^d))$ with $\partial_t u \in L^2(\mathbb{T}^d \times (0,T))$ **solves the HHMHF** if for all $\varphi \in C^\infty(\mathbb{T}^d \times [0,T]; \mathbb{R}^{m+1})$:

$$\int_0^T \int_{\mathbb{T}^d} \partial_t u \cdot \varphi \, dx\, dt + \int_0^T \langle d_{1/2} u, d_{1/2}\varphi \rangle_{\mathrm{od}} \, dt = \int_0^T \int_{\mathbb{T}^d} |d_{1/2} u|^2_{\mathrm{od}} \, u \cdot \varphi \, dx\, dt.$$

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u satisfies the **energy-dissipation inequality** if

$$E_{1/2}(u(t)) + \int_0^t \|\partial_t u(s)\|_{L^2}^2 \, ds \;\le\; E_{1/2}(u^0) \qquad \text{for a.e. } t \in (0,T).$$

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A u satisfying both is called a **weak solution.**

Convergence — Untruncated Scheme

Theorem

Let $u^0 \in H^{1/2}(\mathbb{T}^d;\mathbb{S}^m)$. There exists a subsequence $h_n \to 0$ such that

$$
\begin{aligned}
u^{h_n} &\to u && \text{in } L^2(\mathbb{T}^d \times (0,T)),\\
(-\Delta)^{1/4}(P_{h_n/2} * u^{h_n}) &\rightharpoonup (-\Delta)^{1/4} u && \text{in } L^2(\mathbb{T}^d \times (0,T)),\\
\partial_t^{h_n}(P_{h_n/2} * u^{h_n}) &\rightharpoonup \partial_t u && \text{in } L^2(\mathbb{T}^d \times (0,T)).
\end{aligned}
$$

*The limit u is a **weak solution** and attains u^0 strongly in $H^{1/2}(\mathbb{T}^d)$.*

Here $\partial_t^h v := \dfrac{v(t+h) - v(t)}{h}$ denotes the discrete time derivative.

Convergence — Truncated Scheme

Replace P_h by its Fourier projection onto the first N modes:

$$u^{n+1} = \frac{P_h^N * u^n}{|P_h^N * u^n|}, \qquad \widehat{P_h^N}(k) = e^{-h|k|}\,\chi_{|k|\leq N}.$$

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The condition guarantees $P_h^N > 0$, which is needed for the L^2 convergence of u^h itself.

Proof Idea I: Minimizing Movement

Approximate energy:

$$E_h(u) := \frac{1}{2h} \int_{\mathbb{T}^d} \big(1 - u \cdot (P_h * u)\big)\, dx \xrightarrow{h \to 0} E_{1/2}(u)$$

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Minimizing movement: Each step $u^{n-1} \mapsto u^n$ minimizes [EO15]

$$E_h(u) + \underbrace{\frac{1}{2h}\int_{\mathbb{T}^d} (u - u^{n-1})\cdot\big(P_h * (u - u^{n-1})\big)\,dx}_{=:\,d_h^2(u,u^{n-1})} \quad \text{over } |u| \le 1.$$

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Choosing $u = u^{n-1}$ as competitor and summing over steps:

$$E_h(u^h(T)) + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \left| P_{h/2} * \partial_t^h u^h \right|^2 dx\, dt \;\leq\; E_h(u^0).$$

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Not sharp: the contribution of the metric slope is missing.

Proof Idea II: The De Giorgi Approach

De Giorgi's inequality:

$$E_h(u^h(T)) + \frac{1}{2}\int_0^T \|P_{h/2} * \partial_t^h u^h\|_{L^2}^2 \, dt + \frac{1}{2}\int_0^T |\partial E_h|^2(\tilde{u}^h(t)) \, dt \;\leq\; E_h(u^0)$$

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Metric slope:

$$|\partial E_h|(u) := \limsup_{v \to u} \frac{\big(E_h(u) - E_h(v)\big)^+}{d_h(u,v)}$$

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This approach needs convexity, which is in conflict with our geometric condition $|u|^2 = 1$.

Proof Idea III: Exact Energy-Dissipation Identity

Theorem

With $t_k = kh$, the iterates satisfy

$$E_h(u^h(t_k)) + \frac{1}{2}\int_0^{t_k}\int_{\mathbb{T}^d} \left|P_{h/2} * \partial_t^h u^h\right|^2 + \frac{1}{2}\int_0^{t_k}\int_{\mathbb{T}^d} |P_h * u^h| \left|\partial_t^h u^h\right|^2 = E_h(u^0).$$

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The idea is to show

$$\mathbf{1}_{A_h} \sqrt{|P_h * u^h|}\, \partial_t^h u^h \rightharpoonup \partial_t u \quad \text{in } L^2(\mathbb{T}^d \times (0,T)),$$

where

$$A_h := \bigcup_j O_j \times J_j, \qquad O_j := \bigcap_{nh \in J_j} \left\{ |P_h * u^n| > \tfrac{1}{2} \right\}.$$

Proof Idea IV: Passing to the Limit

From the minimization: the iterates satisfy the **approximative HHMHF**

$$(\mathrm{Id} - u^h \otimes u^h)\big(P_h * \partial_t^h u^h(\cdot - h) + (-\Delta)_h^{1/2} u^h\big) = 0.$$

Proof Idea IV: Passing to the Limit

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Main challenge: Show that this passes to the limit $h \to 0$, yielding the weak formulation of the HHMHF.

Weak-Strong Uniqueness

Theorem

Let u be a weak solution and v a ***strong solution*** *of the HHMHF, i.e. a weak solution with the same initial datum $u^0 \in H^{1/2}(\mathbb{T}^d;\mathbb{S}^m)$ satisfying additionally*

$$d_{1/2}v,\ d_{1/2}\partial_t v \in L^\infty(0,T;L^\infty_{\mathrm{od}}), \qquad \partial_t v \in L^\infty(\mathbb{T}^d \times (0,T)).$$

Then $\mathbf{u} = \mathbf{v}$ a.e. in $\mathbb{T}^d \times (0,T)$.

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Proof idea: Take $F(u-v) := E_{1/2}(u-v) + \frac{1}{2}\|u-v\|^2_{L^2}$, differentiate, throw all derivatives onto v and apply Grönwall.

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Corollary

If a strong solution v exists, the limit of the thresholding scheme is v: $u^{h_n} \to v$.

Thank you for listening!

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