

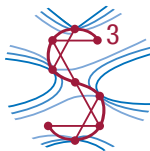
Thresholding Scheme for the Half-Harmonic Map Heat Flow

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joint with Christof Melcher and Endre Süli

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The Half-Harmonic Map Heat Flow

- Half-Dirichlet energy for maps $u : \mathbb{T}^d \rightarrow \mathbb{S}^m$:

$$\begin{aligned} E_{1/2}(u) &= \frac{c_d}{4} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+1}} dy dx = \frac{1}{2} \|(-\Delta)^{1/4} u\|_{L^2}^2 \\ &=: \frac{1}{2} \int_{\mathbb{T}^d} |d_{1/2} u|_{\text{od}}^2 dx.^1 \end{aligned}$$

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- ▶ **Gradient flow** of $E_{1/2}$ on $\mathbb{S}^m$, $u(0) = u^0 : \mathbb{T}^d \rightarrow \mathbb{S}^m$:

$$\partial_t u = -d\pi_{\mathbb{S}^m}(u) \nabla_{L^2} E_{1/2} = -(-\Delta)^{1/2} u + |d_{1/2} u|_{\text{od}}^2 u.$$

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- ▶ $(-\Delta)^{1/2}$ defined via Fourier coefficients:

$$\widehat{(-\Delta)^{1/2} u}(k) = |k| \hat{u}(k), \quad k \in \mathbb{Z}^d.$$

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The Thresholding Scheme

Motivated by Laux–Yip [LY19].

- Solve the **half-heat equation** $\partial_t v + (-\Delta)^{1/2} v = 0$, $v(0) = u^0$. The solution is given by

$$v(t, x) = (P_t * u^0)(x), \quad \text{where } \hat{P}_t(k) = e^{-t|k|}, \quad k \in \mathbb{Z}^d.$$

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- **Algorithm:** Given time step $h > 0$, iterate:

$$u^{n+1} = \frac{P_h * u^n}{|P_h * u^n|}, \quad u^h(t) := \sum_k u^k \chi_{[kh, (k+1)h)}(t).$$

From MCF to Harmonic Map Flows

Flow	$\hat{P}_h(k)$	Ref
MCF (codim. 1)	$e^{-h k ^2}$	[MBO92; LO16]
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Key adaptation: replace the Gaussian $e^{-h|k|^2}$ by the Poisson kernel $e^{-h|k|}$.

Weak Solution

Definition

$u \in L^\infty(0, T; H^{1/2}(\mathbb{T}^d))$ with $\partial_t u \in L^2(\mathbb{T}^d \times (0, T))$ **solves the HHMHF** if for all $\varphi \in C^\infty(\mathbb{T}^d \times [0, T]; \mathbb{R}^{m+1})$:

$$\int_0^T \int_{\mathbb{T}^d} \partial_t u \cdot \varphi \, dx \, dt + \int_0^T \langle d_{1/2} u, d_{1/2} \varphi \rangle_{\text{od}} \, dt = \int_0^T \int_{\mathbb{T}^d} |d_{1/2} u|_{\text{od}}^2 u \cdot \varphi \, dx \, dt.$$

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u satisfies the **energy-dissipation inequality** if

$$E_{1/2}(u(t)) + \int_0^t \|\partial_t u(s)\|_{L^2}^2 \, ds \leq E_{1/2}(u^0) \quad \text{for a.e. } t \in (0, T).$$

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A u satisfying both is called a **weak solution**.

Convergence — Untruncated Scheme

Theorem

Let $u^0 \in H^{1/2}(\mathbb{T}^d; \mathbb{S}^m)$. There exists a subsequence $h_n \rightarrow 0$ such that

$$\begin{aligned} u^{h_n} &\rightharpoonup u && \text{in } L^2(\mathbb{T}^d \times (0, T)), \\ (-\Delta)^{1/4}(P_{h_n/2} * u^{h_n}) &\rightharpoonup (-\Delta)^{1/4}u && \text{in } L^2(\mathbb{T}^d \times (0, T)), \\ \partial_t^{h_n}(P_{h_n/2} * u^{h_n}) &\rightharpoonup \partial_t u && \text{in } L^2(\mathbb{T}^d \times (0, T)). \end{aligned}$$

The limit u is a **weak solution** and attains u^0 strongly in $H^{1/2}(\mathbb{T}^d)$.

Here $\partial_t^h v := \frac{v(t+h) - v(t)}{h}$ denotes the discrete time derivative.

Convergence — Truncated Scheme

Replace P_h by its Fourier projection onto the first N modes:

$$u^{n+1} = \frac{P_h^N * u^n}{|P_h^N * u^n|}, \quad \widehat{P_h^N}(k) = e^{-h|k|} \chi_{|k| \leq N}.$$

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The condition guarantees $P_h^N > 0$, which is needed for the L^2 convergence of u^h itself.

Proof Idea I: Minimizing Movement

Approximate energy:

$$E_h(u) := \frac{1}{2h} \int_{\mathbb{T}^d} (1 - u \cdot (P_h * u)) \, dx \xrightarrow{h \rightarrow 0} E_{1/2}(u)$$

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Choosing $u = u^{n-1}$ as competitor and summing over steps:

$$E_h(u^h(T)) + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} |P_{h/2} * \partial_t^h u^h|^2 \, dx \, dt \leq E_h(u^0).$$

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Not sharp: the contribution of the metric slope is missing.

Proof Idea II: The De Giorgi Approach

De Giorgi's inequality:

$$E_h(u^h(T)) + \frac{1}{2} \int_0^T \|P_{h/2} * \partial_t^h u^h\|_{L^2}^2 dt + \frac{1}{2} \int_0^T |\partial E_h|^2(\tilde{u}^h(t)) dt \leq E_h(u^0)$$

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Metric slope:

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This approach needs convexity, which is in conflict with our geometric condition $|u|^2 = 1$.

Proof Idea III: Exact Energy-Dissipation Identity

Theorem

With $t_k = kh$, the iterates satisfy

$$E_h(u^h(t_k)) + \frac{1}{2} \int_0^{t_k} \int_{\mathbb{T}^d} |P_{h/2} * \partial_t^h u^h|^2 + \frac{1}{2} \int_0^{t_k} \int_{\mathbb{T}^d} |P_h * u^h| |\partial_t^h u^h|^2 = E_h(u^0).$$

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The idea is to show

$$\mathbf{1}_{A_h} \sqrt{|P_h * u^h|} \partial_t^h u^h \rightharpoonup \partial_t u \quad \text{in } L^2(\mathbb{T}^d \times (0, T)),$$

where

$$A_h := \bigcup_j O_j \times J_j, \quad O_j := \bigcap_{nh \in J_j} \{|P_h * u^n| > \tfrac{1}{2}\}.$$

Proof Idea IV: Passing to the Limit

From the minimization: the iterates satisfy the **approximative HHMHF**

$$(\text{Id} - u^h \otimes u^h)(P_h * \partial_t^h u^h(\cdot - h) + (-\Delta)_h^{1/2} u^h) = 0.$$

Proof Idea IV: Passing to the Limit

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Main challenge: Show that this passes to the limit $h \rightarrow 0$, yielding the weak formulation of the HHMHF.

Weak-Strong Uniqueness

Theorem

Let u be a weak solution and v a **strong solution** of the HHMHF, i.e. a weak solution with the same initial datum $u^0 \in H^{1/2}(\mathbb{T}^d; \mathbb{S}^m)$ satisfying additionally

$$d_{1/2}v, d_{1/2}\partial_t v \in L^\infty(0, T; L^\infty_{\text{od}}), \quad \partial_t v \in L^\infty(\mathbb{T}^d \times (0, T)).$$

Then $\mathbf{u} = \mathbf{v}$ a.e. in $\mathbb{T}^d \times (0, T)$.

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Proof idea: Take $F(u - v) := E_{1/2}(u - v) + \frac{1}{2}\|u - v\|_{L^2}^2$, differentiate, throw all derivatives onto v and apply Grönwall.

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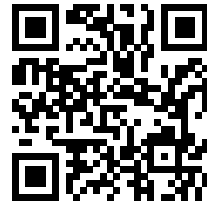
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Corollary

If a strong solution v exists, the limit of the thresholding scheme is v : $u^{h_n} \rightarrow v$.

Thank you for listening!



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