Half-Harmonic Map Heat Flows

Kilian Koch

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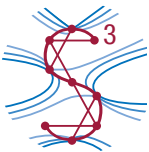

About me

3rd-year PhD student at the Chair of Applied Analysis, RWTH Aachen University



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それから…



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Harmonic map heat flow

Energy for maps $\boldsymbol{u}\colon \mathbb{R}^n \to \mathbb{S}^m$

$$D(\boldsymbol{u}) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \boldsymbol{u}|^2 \, dx$$

J. Eells, J. H. Sampson, Amer. J. Math. 86 (1964)

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$$\partial_t \boldsymbol{u} = -\nabla^{\tan} D(\boldsymbol{u})$$

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$$\partial_t \boldsymbol{u} = -\nabla^{\tan} D(\boldsymbol{u}) = \Delta \boldsymbol{u} + |\nabla \boldsymbol{u}|^2 \boldsymbol{u}$$

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$$\partial_t \boldsymbol{u} = -\nabla^{\tan} D(\boldsymbol{u}) = \Delta \boldsymbol{u} + |\nabla \boldsymbol{u}|^2 \boldsymbol{u}$$

Critical dimension $n = 2$, since $D(\boldsymbol{u}(\lambda\,\cdot)) = \lambda^{2-n} D(\boldsymbol{u})$.

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Blow-up

$$\partial_t \boldsymbol{u} = \Delta \boldsymbol{u} + |\nabla \boldsymbol{u}|^2 \boldsymbol{u} \quad \text{on the unit disk } D$$

Simulation by N. A. Nguyen and A. Reusken, arXiv:2508.20590

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$$\partial_t \boldsymbol{u} = \Delta \boldsymbol{u} + |\nabla \boldsymbol{u}|^2 \boldsymbol{u} \quad \text{on the unit disk } D$$

$$\boldsymbol{u}(t) = \begin{pmatrix} \cos\psi \sin h(r,t) \\ \sin\psi \sin h(r,t) \\ \cos h(r,t) \end{pmatrix}$$

$$h(r,0) = h_0(r), \quad h(1,t) = h_0(1), \quad |h_0(1)| > \pi$$

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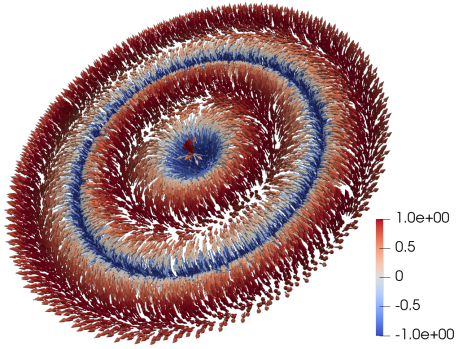

$t = 0.083$

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Half-harmonic map heat flow

Energy for maps $\boldsymbol{u}\colon \mathbb{R}^n \to \mathbb{S}^m$

$$E(\boldsymbol{u}) = \frac{c_n}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\boldsymbol{u}(x) - \boldsymbol{u}(y)|^2}{|x-y|^{n+1}}\, dy\, dx = \frac{1}{2} \|(-\Delta)^{1/4} \boldsymbol{u}\|_{L^2}^2$$

Integral representation: V. Millot, Y. Sire, Arch. Ration. Mech. Anal. 215 (2015)

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$$\begin{aligned} \partial_t \boldsymbol{u} &= -\nabla^{\tan} E(\boldsymbol{u}) \\ &= -(-\Delta)^{1/2} \boldsymbol{u} + \Big(\frac{c_n}{2} \int_{\mathbb{R}^n} \frac{|\boldsymbol{u}(x) - \boldsymbol{u}(y)|^2}{|x-y|^{n+1}}\, dy \Big)\, \boldsymbol{u}(x) \end{aligned}$$

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Half gradient

$$d_{1/2}\boldsymbol{u}(x,y) = \sqrt{\frac{c_n}{2}}\,\frac{\boldsymbol{u}(x)-\boldsymbol{u}(y)}{|x-y|^{1/2}}$$

with the off-diagonal inner product

$$\langle \boldsymbol{F},\boldsymbol{G}\rangle_{\mathrm{od}}(x) = \int_{\mathbb{R}^n} \boldsymbol{F}(x,y)\cdot\boldsymbol{G}(x,y)\,\frac{dy}{|x-y|^n}, \qquad |\boldsymbol{F}|^2_{\mathrm{od}} = \langle \boldsymbol{F},\boldsymbol{F}\rangle_{\mathrm{od}}$$

M. Mazowiecka, A. Schikorra, J. Funct. Anal. 275 (2018); flow in this form: J. Wettstein, Nonlinear Anal. 214 (2022)

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Hence the flow reads

$$\partial_t \boldsymbol{u} + (-\Delta)^{1/2}\boldsymbol{u} = |d_{1/2}\boldsymbol{u}|^2_{\mathrm{od}}\,\boldsymbol{u}.$$

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Shrinkers and expanders

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Shrinker $\boldsymbol{u}(x,t) = \boldsymbol{U}\left(\frac{x}{T-t}\right)$ $(-\Delta)^{1/2}\boldsymbol{U}(y) + y \cdot \nabla \boldsymbol{U}(y) = |d_{1/2}\boldsymbol{U}|^2_{\mathrm{od}}(y)\,\boldsymbol{U}(y)$

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Expander $\boldsymbol{u}(x,t) = \boldsymbol{U}\left(\frac{x}{t}\right)$ $(-\Delta)^{1/2}\boldsymbol{U}(y) - y \cdot \nabla \boldsymbol{U}(y) = |d_{1/2}\boldsymbol{U}|^2_{\mathrm{od}}(y)\,\boldsymbol{U}(y)$

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The counterpart to a shrinker is the expander.

Global Struwe type solutions

Theorem (Wettstein, Struwe)

For any $\boldsymbol{u}_0 \in H^{1/2}(\mathbb{S}^1; N)$ *there exists a weak solution* $\boldsymbol{u}\colon [0,\infty) \times \mathbb{S}^1 \to N$ *with non-increasing* $1/2$*–Dirichlet energy, smooth on* $(0,\infty) \times \mathbb{S}^1$ *except in finitely many points* (T_k, x_k), $T_1 \leq \cdots \leq T_K$.

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Regularity:

- $\boldsymbol{u} \in L^\infty_t H^{1/2}_x \cap H^1_t L^2_x\big([0,\infty) \times \mathbb{S}^1\big)$
- $\boldsymbol{u} \in H^1_{\mathrm{loc}}\big([0,T_1) \times \mathbb{S}^1\big) \subset C^0\big([0,T_1); H^{1/2}\big)$

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Implicitly contains local $H^{1/2}$ well-posedness.

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Infinite-time blow-up for $\mathbb{R} \to \mathbb{S}^1$ was shown by Sire–Wei–Zheng.

J. Wettstein, Math. Eng. 5 (2023); M. Struwe, Anal. PDE 17 (2024); Y. Sire, J. Wei, Y. Zheng, Amer. J. Math. 143 (2021)

Finding expanders

Homogeneous initial data $\boldsymbol{u}_0(\lambda x) = \boldsymbol{u}_0(x)$ for all $\lambda > 0$

Expanders for HMHF and LLG: P. Germain, M. Rupflin, Ann. IHP Anal. Non Linéaire 28 (2011); S. Gutiérrez, A. de Laire, Nonlinearity 32 (2019)

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Simulated example with jump:

$$\boldsymbol{u}_0(x) \approx \boldsymbol{e}_2\, \chi_{(-\infty,0)}(x) - \boldsymbol{e}_2\, \chi_{(0,\infty)}(x)$$

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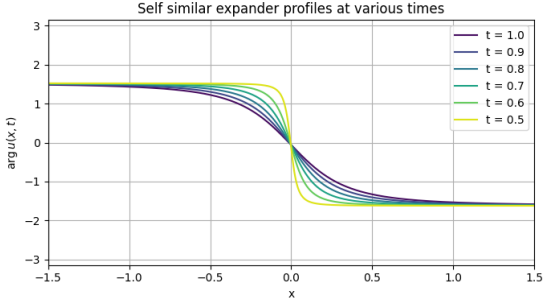
Self-similar $\mathbb{S}^1$-valued expander:

$$\boldsymbol{u}(x,t) = \boldsymbol{U}\left(\tfrac{x}{t}\right)$$



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Existence

We identified an initial data space $\mathcal{A}$ containing $L^\infty \cap \mathrm{BV}(\mathbb{R})$, $L^\infty \cap \dot{B}^{\frac{n}{2}}_{2,\infty}(\mathbb{R}^n)$, respectively, and a solution space X_T such that:

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Theorem

There exists $\varepsilon > 0$ such that for all $\boldsymbol{a}\colon \mathbb{R}^n \to \mathbb{S}^m$ with $\boldsymbol{a} \in \mathcal{A}$ and $[\boldsymbol{a}]_{\mathcal{A}_T} < \varepsilon$, there exists $\boldsymbol{u} \in X_T$ solving

$$\begin{cases} \partial_t \boldsymbol{u} + (-\Delta)^{1/2} \boldsymbol{u} = |d_{1/2}\boldsymbol{u}|^2_{\mathrm{od}}\, \boldsymbol{u}, \\ \boldsymbol{u}(0) = \boldsymbol{a}. \end{cases}$$

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$L^\infty \cap \dot{H}^{1/2}(\mathbb{R}) \subsetneq \mathcal{A}$.

K. Koch, C. Melcher, arXiv:2504.06933

Initial data space $\mathcal{A}$

Harmonic extension $\boldsymbol{v}(t) = p_t * \boldsymbol{a}$: $\quad \partial_t \boldsymbol{v} + (-\Delta)^{1/2} \boldsymbol{v} = 0, \ \boldsymbol{v}(0) = \boldsymbol{a}$

$$\mathcal{M}_T \boldsymbol{a}(x) := \sup_{0<r<T} \Big(\int_0^r \fint_{B_r(x)} |d_{1/2} \boldsymbol{v}|_{\mathrm{od}}^2 \, dy \, dt \Big)^{1/2}$$

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Harmonic map heat flow (analogous to C. Wang) $\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} = 0,\ \boldsymbol{u}(0) = \boldsymbol{a}$

$$\sup_{0<r<\sqrt{T}} \sup_{x\in\mathbb{R}^n} \int_0^{r^2} \fint_{B_r(x)} |\nabla \boldsymbol{u}|^2\, dy\, dt$$

$T = \infty$: equivalent to $\|\boldsymbol{a}\|^2_{\mathrm{BMO}}$ (Carleson characterization)

H. Koch, D. Tataru, Adv. Math. 157 (2001); maximal functions: C. Fefferman, E. M. Stein, Acta Math. 129 (1972); C. Wang, Arch. Ration. Mech. Anal. 200 (2011); Carleson characterization: E. M. Stein, Harmonic Analysis (1993)

Uniqueness

Theorem

There exists $\delta > 0$ such that solutions $\boldsymbol{u}_1, \boldsymbol{u}_2 \in X_T$ with $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) \in \mathcal{A}$ and

$$\limsup_{R \searrow 0} [\boldsymbol{u}_i]_{X_R} \leq \delta, \qquad i = 1, 2,$$

satisfy $\boldsymbol{u}_1 \equiv \boldsymbol{u}_2$.

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Self-similar expanders

Theorem (Self-similar expanders)

Let $\boldsymbol{a}(x) = \varphi\left(\frac{x}{|x|}\right)$ *with* $\varphi\colon \mathbb{S}^{n-1} \to \mathbb{S}^m$ *Lipschitz. If* $\operatorname{Lip}(\varphi) \ll 1$*, then* $\boldsymbol{a} \in \mathcal{A}$ *with* $[\boldsymbol{a}]_{\mathcal{A}} \lesssim \operatorname{Lip}(\varphi)$*, and there exists a self-similar expander*

$$\boldsymbol{u}(x,t) = \boldsymbol{U}\left(\tfrac{x}{t}\right) \qquad \text{with} \qquad \boldsymbol{u}(0) = \boldsymbol{a}.$$

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$$\boldsymbol{u}(x,t) = \boldsymbol{U}\big(\tfrac{x}{t}\big) \qquad \text{with} \qquad \boldsymbol{u}(0) = \boldsymbol{a}.$$

Proof The scaling of the half-harmonic map heat flow is $\boldsymbol{u}(\lambda x, \lambda t)$, and $\boldsymbol{a}(\lambda\,\cdot) = \boldsymbol{a}$. By uniqueness and choosing $\lambda = 1/t$, one obtains

$$\boldsymbol{u}(x,t) = \boldsymbol{u}\big(\tfrac{x}{t}, 1\big), \qquad \text{i.e.} \qquad \boldsymbol{U}(y) = \boldsymbol{u}(y,1).$$

K. Koch, C. Melcher, arXiv:2504.06933

The Thresholding Scheme

Motivated by Laux–Yip.

- Solve the **half-heat equation** $\partial_t \boldsymbol{v} + (-\Delta)^{1/2}\boldsymbol{v} = 0$, $\boldsymbol{v}(0) = \boldsymbol{u}^0$. The solution is given by

$$\boldsymbol{v}(t,x) = (P_t * \boldsymbol{u}^0)(x), \quad \text{where } \hat{P}_t(k) = e^{-t|k|},\ k \in \mathbb{Z}^d.$$

T. Laux, N. K. Yip, Arch. Ration. Mech. Anal. (2019)

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- Project pointwise onto $\mathbb{S}^m$:

$$\boldsymbol{u}(t,x) = \frac{\boldsymbol{v}(t,x)}{|\boldsymbol{v}(t,x)|}.$$

- **Algorithm:** Given time step $h > 0$, iterate:

$$\boldsymbol{u}^{n+1} = \frac{P_h * \boldsymbol{u}^n}{|P_h * \boldsymbol{u}^n|}, \qquad \boldsymbol{u}^h(t) := \sum_k \boldsymbol{u}^k\, \chi_{[kh,\,(k+1)h)}(t).$$

T. Laux, N. K. Yip, Arch. Ration. Mech. Anal. (2019)

Minimizing Movement

Approximate energy:

$$E_h(\boldsymbol{u}) := \frac{1}{2h} \int_{\mathbb{T}^d} \left(1 - \boldsymbol{u} \cdot (P_h * \boldsymbol{u})\right) dx \xrightarrow{h \to 0} E(\boldsymbol{u})$$

Minimizing Movement

Approximate energy:

$$E_h(\boldsymbol{u}) := \frac{1}{2h}\int_{\mathbb{T}^d}\big(1-\boldsymbol{u}\cdot(P_h*\boldsymbol{u})\big)\,dx \xrightarrow{h\to 0} E(\boldsymbol{u})$$

Minimizing movement: Each step $\boldsymbol{u}^{n-1}\mapsto\boldsymbol{u}^n$ minimizes

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Choosing $\boldsymbol{u}=\boldsymbol{u}^{n-1}$ as competitor and summing over steps:

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Not sharp: the contribution of the metric slope is missing.

S. Esedoğlu, F. Otto, Comm. Pure Appl. Math. 68 (2015)

Exact Energy-Dissipation Identity

Theorem

With $t_k = kh$, the iterates satisfy

$$E_h(\boldsymbol{u}^h(t_k)) + \frac{1}{2}\int_0^{t_k}\int_{\mathbb{T}^d} \left|P_{h/2} * \partial_t^h \boldsymbol{u}^h\right|^2 + \frac{1}{2}\int_0^{t_k}\int_{\mathbb{T}^d} \left|P_h * \boldsymbol{u}^h\right| \left|\partial_t^h \boldsymbol{u}^h\right|^2 = E_h(\boldsymbol{u}^0).$$

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In the limit $h \to 0$ we obtain

$$E(\boldsymbol{u}(t)) + \int_0^t \|\partial_t \boldsymbol{u}(s)\|_{L^2}^2\, ds \;\leq\; E(\boldsymbol{u}^0).$$

Weak Solution

Definition

$\boldsymbol{u} \in L^\infty(0,T;\, H^{1/2}(\mathbb{T}^d))$ with $\partial_t \boldsymbol{u} \in L^2(\mathbb{T}^d \times (0,T))$ **solves the HHMHF** if for all $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^d \times [0,T];\, \mathbb{R}^{m+1})$:

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A $\boldsymbol{u}$ satisfying both is called a **weak solution**.

Convergence and weak–strong uniqueness

Theorem

Let $\boldsymbol{u}^0 \in H^{1/2}(\mathbb{T}^d; \mathbb{S}^m)$.

- *For a subsequence* $h_n \to 0$: $\boldsymbol{u}^{h_n} \to \boldsymbol{u}$ *in* $L^2(\mathbb{T}^d \times (0,T))$, *and* $\boldsymbol{u}$ *is a* ***weak solution*** *attaining* $\boldsymbol{u}^0$ *strongly in* $H^{1/2}(\mathbb{T}^d)$.

K. Koch, C. Melcher, E. Süli, arXiv:2609.25912

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- ***Uniqueness*** *among weak solutions as long as a* ***strong solution*** $\boldsymbol{v}$ *with* $\boldsymbol{v}(0) = \boldsymbol{u}^0$ *exists.*

K. Koch, C. Melcher, E. Süli, arXiv:2609.25912

Convergence rates

Theorem (Uniform proximity to $\mathbb{S}^m$)

Let $\boldsymbol{u}^0 \in C^{0,1}(\mathbb{T}^d; \mathbb{S}^m)$ and $T > 0$ small enough, depending on $[\boldsymbol{u}^0]_{C^{0,1}}$. Then for $h > 0$ small enough

$$\operatorname{dist}\left((P_h * \boldsymbol{u}^n)(x), \mathbb{S}^m\right) \lesssim h \qquad \text{for all } nh \leq T,\ x \in \mathbb{T}^d.$$

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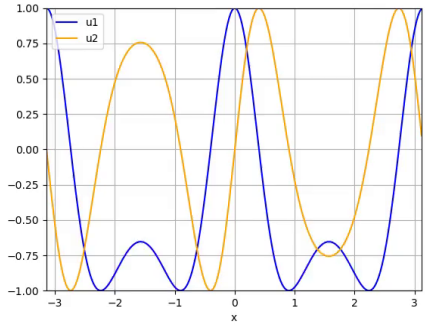
Theorem (Fully discrete convergence)

*If there exists a solution $\boldsymbol{u} \in C^\infty([0,T]\times\mathbb{T}^d)$ of the HHMHF, the **truncated** scheme converges under the restriction $N \gtrsim \frac{\log(1/h)}{h}$:*

$$\|\boldsymbol{u}_N^n - \boldsymbol{u}(nh)\|_{L^2} \lesssim h.$$

K. Koch, C. Melcher, N. A. Nguyen, A. Reusken, in preparation

Simulation



$$\boldsymbol{u}_0(x) = \begin{pmatrix} \cos(4\sin x) \\ \sin(4\sin x) \end{pmatrix}, \qquad T = 5, \quad h = 10^{-2}, \quad N = 2^8$$

Simulation by N. A. Nguyen

Outlook

Shrinkers in higher dimensions $n > 1 \quad \boldsymbol{u}(x,t) = \boldsymbol{U}\left(\frac{x}{T-t}\right)$

Blow-up: J. Wettstein, Math. Eng. 5 (2023); M. Struwe, Anal. PDE 17 (2024); stochastic HMHF: A. Hocquet, J. Evol. Equ. 18 (2018)

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Blow-up in the critical dimension $n = 1$ existence or non-existence of (x_0, T) with

$$\limsup_{t \nearrow T} \int_{B_R(x_0)} |(-\Delta)^{1/4} \boldsymbol{u}(t)|^2 \, dx \geq \varepsilon_1 \qquad \text{for all } R > 0$$

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Stochastic perturbation

$$d\boldsymbol{u} + (-\Delta)^{1/2} \boldsymbol{u} \, dt = |d_{1/2}\boldsymbol{u}|^2_{\mathrm{od}} \, \boldsymbol{u} \, dt + \boldsymbol{u} \times \circ \, d\boldsymbol{W}$$

Blow-up: J. Wettstein, Math. Eng. 5 (2023); M. Struwe, Anal. PDE 17 (2024); stochastic HMHF: A. Hocquet, J. Evol. Equ. 18 (2018)

Thank you for listening!



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