A Thresholding Scheme for the Half-Harmonic Map Heat Flow

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Outline

The Problem and History

The Thresholding Scheme

Main Result

Proof Idea

Weak-Strong Uniqueness

The Half-Harmonic Map Heat Flow

- Half-Dirichlet energy for maps $u : \mathbb{T}^d \to \mathbb{S}^m$:

$$E_{1/2}(u) = \frac{\gamma_d}{2} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{|u(x+h) - u(x)|^2}{|h|^{d+1}} \, dh \, dx =: \frac{1}{2} \int_{\mathbb{T}^d} |d_{1/2} u|_{\mathrm{od}}^2 \, dx.^1$$

[1] Notation following Mazowiecka–Schikorra [MS18].

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- **Gradient flow** of $E_{1/2}$ on $\mathbb{S}^m$, $u(0) = u_0 : \mathbb{T}^d \to \mathbb{S}^m$:

$$\partial_t u = -d\pi_{\mathbb{S}^m}(u) \nabla_{L^2} E_{1/2} = -(-\Delta)^{1/2} u + |d_{1/2} u|^2_{\mathrm{od}} \, u.$$

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- $(-\Delta)^{1/2}$ defined via Fourier coefficients:

$$\widehat{(-\Delta)^{1/2} u}(k) = |k| \, \hat{u}(k), \quad k \in \mathbb{Z}^d.$$

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History of the Half-Harmonic Map Heat Flow

Da Lio & Rivière (2011) Introduction of $\frac{1}{2}$-harmonic maps [DR11]

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Koch & Melcher (2025)	Well-posedness for rough data [KM25]

The Thresholding Scheme

Motivated by Laux–Yip [LY19].

- Solve the **half-heat equation** $\partial_t v + (-\Delta)^{1/2} v = 0$, $v(0) = u_0$. The solution is given by

$$v(t, x) = (P_t * u_0)(x), \quad \text{where } \hat{P}_t(k) = e^{-t|k|},\ k \in \mathbb{Z}^d.$$

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- **Algorithm.** Given time step $h > 0$, set $u^0 = u_0$ and iterate:

$$u^{n+1} = \frac{P_h * u^n}{|P_h * u^n|}, \qquad u^h(t) := \sum_{k=0}^{M-1} u^k \, \chi_{[kh,\,(k+1)h)}(t).$$

From MCF to Harmonic Map Flows

Flow	$\hat{P}_h(k)$	Ref
MCF (codim. 1)	$e^{-h\|k\|^2}$	[LO16]
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Key adaptation: replace the Gaussian $e^{-h|k|^2}$ by the Poisson kernel $e^{-h|k|}$.

Weak Solution

Definition

$u \in L^\infty(0,T;H^{1/2}(\mathbb{T}^d))$ with $\partial_t u \in L^2(\mathbb{T}^d \times (0,T))$ **solves the HHMHF** if for all $\varphi \in C^\infty(\mathbb{T}^d \times [0,T];\mathbb{R}^{m+1})$:

$$\int_0^T \int_{\mathbb{T}^d} \partial_t u \cdot \varphi \, dx\, dt + \int_0^T \langle d_{1/2} u, d_{1/2}\varphi \rangle_{\mathrm{od}} \, dt = \int_0^T \int_{\mathbb{T}^d} |d_{1/2} u|^2_{\mathrm{od}} \, u \cdot \varphi \, dx\, dt.$$

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u satisfies the **energy inequality** if

$$E_{1/2}(u(t)) + \int_0^t \|\partial_t u(s)\|^2_{L^2} \, ds \;\leq\; E_{1/2}(u^0) \qquad \text{for a.e. } t \in (0,T).$$

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A u satisfying both is called a **weak solution.**

Convergence — Untruncated Scheme

Theorem

Let $u^0 \in H^{1/2}(\mathbb{T}^d;\mathbb{S}^m)$. There exists a subsequence $h_n \to 0$ such that

$$
\begin{aligned}
u^{h_n} &\to u && \text{in } L^2(\mathbb{T}^d \times (0,T)),\\
(-\Delta)^{1/4}(P_{h_n/2} * u^{h_n}) &\rightharpoonup (-\Delta)^{1/4} u && \text{in } L^2(\mathbb{T}^d \times (0,T)),\\
\partial_t^{h_n}(P_{h_n/2} * u^{h_n}) &\rightharpoonup \partial_t u && \text{in } L^2(\mathbb{T}^d \times (0,T)).
\end{aligned}
$$

*The limit u **solves the HHMHF**, attaining u^0 in $H^{1/2}(\mathbb{T}^d)$.*

Here $\partial_t^h v := \dfrac{v(t+h)-v(t)}{h}$ denotes the discrete time derivative.

Convergence — Truncated Scheme

Replace P_h by its Fourier projection onto the first N modes:

$$u^{n+1} = \frac{P_h^N * u^n}{|P_h^N * u^n|}, \qquad \widehat{P_h^N}(k) = e^{-h|k|}\,\chi_{|k|\le N}.$$

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$$\textit{i.e. qualitatively } N \gtrsim \frac{\log(1/h)}{h}.$$

Proof Idea I: Minimizing Movement

Approximate energy:

$$E_h(u) := \frac{1}{2h} \int_{\mathbb{T}^d} \big(1 - u \cdot (P_h * u)\big)\, dx \xrightarrow{h \to 0} E_{1/2}(u)$$

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Minimizing movement: Each step $u^{n-1} \mapsto u^n$ minimizes

$$E_h(u) + \frac{1}{2h} \int_{\mathbb{T}^d} (u - u^{n-1}) \cdot \big(P_h * (u - u^{n-1})\big)\, dx \quad \text{over } |u| \le 1.$$

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Choosing $u = u^{n-1}$ as competitor and summing over steps:

$$E_h(u^h(T)) + \frac{1}{2}\int_0^T\int_{\mathbb{T}^d} \left|P_{h/2} * \partial_t^h u^h\right|^2 dx\,dt \;\le\; E_h(u^0).$$

Proof Idea II: Passing to the Limit

From the minimization: the iterates satisfy the **approximative HHMHF**

$$(\mathrm{Id} - u^h \otimes u^h)\big(P_h * \partial_t^h u^h(\cdot - h) + (-\Delta)_h^{1/2} u^h\big) = 0.$$

Proof Idea II: Passing to the Limit

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Main challenge: Show that this passes to the limit $h \to 0$, yielding the weak formulation of the HHMHF.

Weak-Strong Uniqueness

Theorem

*Let u be a weak solution and v a **strong solution** of the HHMHF, i.e. a weak solution with the same initial datum $u_0 \in H^{1/2}(\mathbb{T}^d; \mathbb{S}^m)$ satisfying additionally*

$$\partial_t v,\ |d_{1/2} v|_{\mathrm{od}},\ |d_{1/2} \partial_t v|_{\mathrm{od}} \in L^\infty((0,T) \times \mathbb{T}^d).$$

Then $\mathbf{u} = \mathbf{v}$ a.e. in $\mathbb{T}^d \times (0,T)$.

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Then $\mathbf{u} = \mathbf{v}$ a.e. in $\mathbb{T}^d \times (0,T)$.

Corollary

If a strong solution v exists and the limit u from the previous theorem satisfies the energy inequality, then $u^{h_n} \to v$.

Thank you for listening!

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